Generalized Inverse Example

by
Harvey J. Greenberg
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Let $A = [1\ 2\ 3]$ and consider $A^+ = [\frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}]'$ Then, $AA^+ = [1]$, so $AA^+A = A$, which implies A^+ is a generalized inverse of A. The equation $Ax = b$ is $x_1 + 2x_2 + 3x_3 = b$, and $A^+b = [\frac{b}{6},\ \frac{b}{6},\ \frac{b}{6}]'$, so this is one solution (with $y = 0$). We have:

$$A^+A = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}, \quad \text{so} \quad I - A^+A = \frac{1}{6} \begin{bmatrix} 5 & -2 & -3 \\ -1 & 4 & -3 \\ -1 & -2 & 3 \end{bmatrix}.$$

The theorem then says the solutions are of the form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} b \\ b \\ b \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 5y_1 - 2y_2 - 3y_3 \\ -y_1 + 4y_2 - 3y_3 \\ -y_1 - 2y_2 + 3y_3 \end{bmatrix}.$$

For example, another solution is $x = [\frac{b+5}{6},\ \frac{b-1}{6},\ \frac{b-1}{6}]'$ for $y = (1, 0, 0)$.